Zeta Functions

Consider polynomials $f_1, \dots, f_r \in \mathbb{F}_p[x_1, \dots, x_m]$. We want to find

$$\nu_n = \#\{(a_1, \dots, a_m) \in \mathbb{F}_{p^n}^m \mid f_1(\underline{a}) = \dots = f_r(\underline{a}) = 0\}.$$

Moreover, what does the sequence $\{\nu_n\}$ do? Let X be a scheme of fin. type $/\mathbb{Z}$. Let $|X|$ be the closed points, and for $x \in |X|$, $N(x) = |k(x)|$.

Def: The Hasse-Weil zeta function is $\zeta_X(s) = \prod_{x \in |X|} \frac{1}{1 - N(x)^{-s}}$, this converges for $Re(s) \gg 0$.

As an example, if $X = \operatorname{Spec} \mathbb{Z}$, $\zeta_X(s) = \prod_{p \in \mathbb{Z}} \frac{1}{1 - p^{-s}} = \prod_{p \in \mathbb{Z}} (1 + p^{-s} + p^{-2s} + \dots) = \sum n^{-s}$ – the Riemann zeta function.

We will consider those X which are of fin. type $/\mathbb{F}_q$, $q = p^n$. Then for any $x \in |X|$, $[k(x) : \mathbb{F}_q] = \deg(x)$, so $N(x) = q^{\deg(x)}$, and set $t = q^{-s}$.

Def: $Z(X, t) = \prod_{x \in |X|} \frac{1}{1 - t^{\deg(x)}}$

Conj (Weil): X sm. proj. $/\mathbb{F}_q$.

1) $Z(X, t)$ is a rational function of the form:

$$\frac{P_1 \cdot \dots \cdot P_{2d-1}}{P_0 \cdots P_{2d}},$$

$d = \dim X$, $P_i \in K[t]$, char $k = 0$.

2) $P_0 = 1 - t$, $P_{2d} = 1 - q^d t$, and $P_r = \prod_i (1 - a_{r,i} t)$, $a_{r,i}$ are alg. integers.

3) $Z(1/q^d t) = \pm q^{d\chi/2} t^{\chi} Z(t)$, χ = euler char of X = $\Delta \cdot \Delta$.

4) (Riemann Hypothesis) $\forall a_{r,i}$ and all conjugates under the Galois group have abs. value $q^{r/2}$.

5) If X is a specialization of sm. proj. $Y/\#$ field, then $\deg P_r = b_r(Y(\mathbb{C}))$.

Thm (Grothendieck): Z_X is rational if X is of finite type $/\mathbb{F}_q$.